Graham Pollak Theorem

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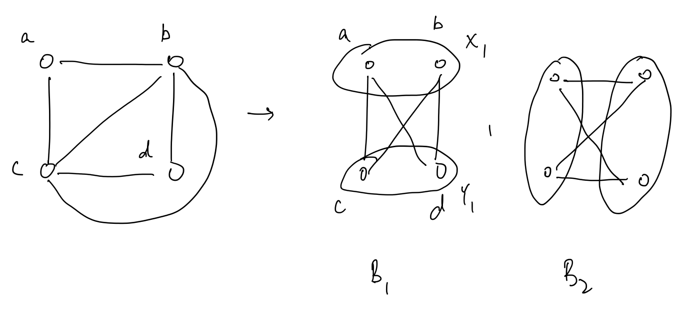
Given: $G = (V, E)$ is a complete graph on n vertices. (will also use K_n)
T complete b.p. graphs
$B_1 = (V_1 = (X_1, Y_1), E_1), \ldots, B_T = (V_T = (X_T, Y_T), E_T)$
where each $V_i \subseteq V$ (& $E_i = X_i \times Y_i$)

Then $B_1, \ldots, B_T$ **cover** G if each edge $e \in E$ occurs in at least one E_i.

Example: G:



B_1 & B_2 cover G

what if $n = 5$, $n = 6$? Are two bipartite graphs enough to cover G?

How many do you need?

Claim: $\lceil \log_2 n \rceil$ bipartite graphs are necessary & sufficient to cover G.

Proof: Necessary: Suppose T complete b.p. graphs $B_1, \ldots, B_T$ cover G. For each $v \in V$, we assign a bit string $s_1^v \ldots s_T^v$ as follows:

$s_t^v = 1$ if $v \in X_t$ (i.e., in the left bipartition in the t^{th} b.p. graph)
$= 0$ o.w.

Note that if two vertices v, w have the same bit string, then they are never in different bipartitions & the edge between them does not appear in any $B_1, \ldots, B_T$.
It follows that the bit string must have length at least $\lceil \log_2 n \rceil$, and hence $T \geq \lceil \log_2 n \rceil$

The proof of sufficiency is now immediate. ∎

Now instead of a cover, we want bipartite graphs that partition the set of edges.

Given $G = (V, E) = K_n$, the complete b.p. graphs $B_1 = (V_1 = (X_1, Y_1), E_1), \ldots, B_T (V_T = (X_T, Y_T), E_T)$ s.t. $E_t = X_t \times Y_t$ partition G if each edge occurs in exactly one b.p. graph.

How many b.p. graphs are enough to partition K_n?

Claim: $(n-1)$ b.p. graphs enough.
Proof: Number the vertices. Then
$$B_i = ((X_i = i, Y_i = \{i+1, \ldots, n\}), E_i = X_i \times Y_i)$$

GRAHAM - POLLAK THEOREM: $(n-1)$ b.p. graphs are needed to partition K_n.

Suppose $B_1, \ldots, B_T$ partition K_n. $T < n-1$.
Let's put variable x_i on vertex i of K_n.
Now since each edge appears exactly once,
$$2 \sum_{e=(i,j)} x_i x_j = 2 \sum_{t=1}^{T} \Big(\sum_{i \in X_t} x_i\Big)\Big(\sum_{i \in Y_t} x_i\Big)$$

add $\sum_i x_i^2$ to both sides. Then:
$$\Big(\sum_i x_i\Big)^2 = \sum_i x_i^2 + 2 \sum_{t=1}^{T} \Big(\sum_{i \in X_t} x_i\Big)\Big(\sum_{i \in Y_t} x_i\Big)$$

(note that this is satisfied for all assignment of values to $x_1, \ldots, x_n$)

Now, suppose we choose $x_1, \ldots, x_n$ to satisfy:
$$\sum_i x_i = 0$$
for each b.p graph $G_t = ((X_t \cup Y_t), E_t)$
$$\sum_{i \in X_t} x_i = 0 \qquad (T < n-1)$$

These are $\leq n-1$ linear equalities

$$A = \underbrace{\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & 0 & 0 \cdots & 0 & 1 \\ 1 & 0 & 1 & 1 \cdots & 0 & 0 \\ \vdots & & & & & \\ 0 & 0 & 1 & 0 \cdots & 1 & 1 \end{bmatrix}}_{n} x = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \quad (\leq n-1 \text{ rows})$$

Thus, there must exist a non-trivial soln.
i.e., $\exists x^*$ that satisfies these equalities, & $x^* \neq 0$.
(this is only true if $\leq n-1$ rows!)
But then:
$$\Big(\sum_i x_i^*\Big)^2 = \sum_i x_i^{*2} + 2 \sum_{t=1}^{T} \Big(\sum_{i \in X_t} x_i^*\Big)\Big(\sum_{i \in Y_t} x_i^*\Big)$$
$$\downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow$$
$$0 \qquad\qquad \neq 0 \qquad\qquad 0$$

gives a contradiction.
Hence, $T \geq n-1$ ∎

One can give a weaker bound by a different technique.

Recall the adjacency matrix for an n-vertex graph has size $n \times n$, with '1' in its (i,j) entry if the edge $\{i,j\}$ exists.

Claim: The adjacency matrix for K_n has rank $n-1$.

Proof: Let A_n be the adjacency matrix for K_n. Then $A_n + I_n = \mathbb{1}_n$, the proof follows. ∎
(I_n: identity matrix, rank n; $\mathbb{1}_n$: all ones matrix, rank 1.)

Claim: The adjacency matrix for any complete bipartite graph on n vertices has rank ≤ 2.

Proof: Let $G = (X \cup Y, X \times Y)$ be a complete b.p. graph. Let $\delta(X), \delta(Y)$ be the characteristic vectors for the left & right partitions (i.e., $\delta(X) \in \{0,1\}^n$, & $\delta_v(X) = 1$ if $v \in X$). Then the adjacency matrix for the b.p. graph is exactly
$$\delta(X)\,\delta(Y)^T + \delta(Y)\,\delta(X)^T$$
& both of these terms have rank 1.

Theorem: Suppose the complete b.p. graphs $B_1 = (X_1 \cup Y_1, X_1 \times Y_1), \ldots, B_T = (X_T \cup Y_T, X_T \times Y_T)$ partition K_n. Then $T \geq \lceil (n-1)/2 \rceil$.
(note that this is a weaker bound than the Graham-Pollak theorem).

Proof: Let A be the adjacency matrix for K_n, & $B_1, \ldots, B_T$ be the adjacency matrices for the bipartite matrices. Since they partition the edge set in K_n,
$$A = B_1 + \ldots + B_T$$
(A: rank $n-1$; each B_t: rank 2)

Since the rank of the sum of matrices is at most the sum of the ranks, the theorem follows. ∎